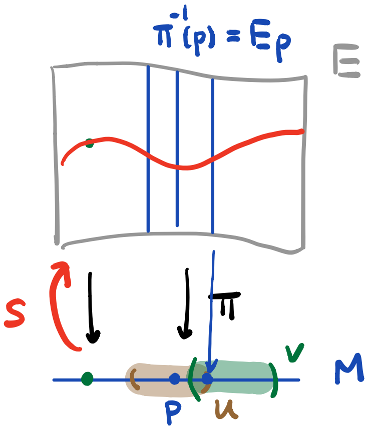
MATH 5061 Lecture on 2/19/2020

Announcement: Problem Set 2 due on Mar 4.

Last time ...... Vector Bundles $\pi : E \longrightarrow M$



local model: $\mathcal{U} \times \mathbb{R}^q$ ← linear structure

open subsets

vector space $\sim E_p$

Alternatively: $g_{UV} : U \cap V \longrightarrow GL(\mathbb{R}^q)$

- $g_{UU} \equiv id$
- $g_{UV}\, g_{VZ}\, g_{ZU} = id$ (cocycle condition)

$(M, \{U_\alpha\}, \{g_{UV}\}) \longleftrightarrow$ vector bundle

- Sections: $\Gamma(E) := \{ s : M \to E \mid \pi \circ s = id_M \}$
- New Vector Bundles: $E, E^*, E_1 \oplus E_2, E_1 \otimes E_2$ etc...
- Useful Examples:

$E =$	TM	T^*M	T^r_sM	$\Lambda^k T^*M$
$\Gamma(E) =$	vector fields	1-forms	(r,s)-tensors	k-forms

Question: How to recognize a tensor field?

For simplicity, consider $(0,2)$-tensors:

(1) Given $(0,2)$-tensor $\alpha \in \Gamma(T^0_2M)$

- At each p, get bilinear $\alpha_p : T_pM \times T_pM \longrightarrow \mathbb{R}$
- Given $X, Y \in \mathfrak{X}(M) := \Gamma(TM)$, we define $f : M \to \mathbb{R}$

$$f(p) := \alpha_p(X_p, Y_p) \in \mathbb{R}$$

$$\alpha \rightsquigarrow \underbrace{\begin{matrix} \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow C^\infty(M) \\ (X, Y) \longmapsto f \end{matrix}}_{Q:\text{ Given this, does it define a tensor?}} \quad (*)$$

← bilinear / $C^\infty(M)$!

(2) Given (#) : $\mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow C^\infty(M)$ which is bilinear / $C^\infty(M)$

Want: At each $p \in M$, define bilinear $\alpha_p : T_pM \times T_pM \to \mathbb{R}$.

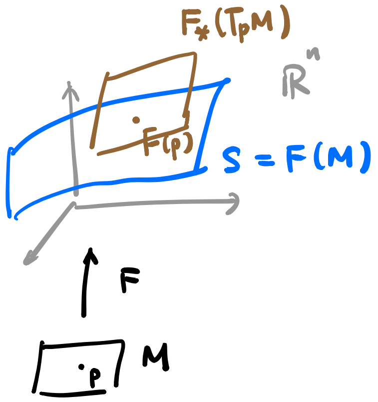
Fix p, and $X_p, Y_p \in T_pM$ ⇝ extend X_p, Y_p to vector fields $X, Y \in \mathfrak{X}(M)$

then $(X, Y) \xmapsto{(\#)} f \in C^\infty(M)$, next $f(p) =: \alpha_p(X_p, Y_p)$

Check: $f(p)$ is indep. of the extensions X, Y. (Pf: last time)

Example (Submanifold geometry in $\mathbb{R}^n$)

- $F : M^n \longrightarrow \mathbb{R}^n$, $n > m$, immersed submanifold.



tangent bundle (intrinsic)

$$\begin{tikzcd} TM := \coprod_{p \in M} T_pM \arrow[d, "\pi"'] \\ M \end{tikzcd}$$

normal bundle (extrinsic)

$$\begin{tikzcd} NM := \coprod_{p \in M} (F_*(T_pM))^\perp \arrow[d, "\pi"'] \\ M \end{tikzcd}$$

- 1st f.f.: g : (0,2)-tensor, symmetric, pos. def.

$X, Y \in \mathfrak{X}(M)$ $\qquad g(X,Y)(p) := \langle F_*X, F_*Y \rangle_{\mathbb{R}^n}(p)$

- 2nd f.f.: h : (0,2)-(vector-valued) tensor, symmetric

$X, Y \in \mathfrak{X}(M)$ $\qquad h(X,Y) := \underbrace{\left(D_{F_*X} F_*Y\right)^\perp}_{\in\, \Gamma(NM)}$

here D is the std. derivative in $\mathbb{R}^n$

- $m = n-1$ (hypersurface case)

Fix some (global) unit normal $\nu : M \to \mathbb{R}^n$.

define: $h(X,Y) = \langle D_{F_*X} F_*Y, \nu \rangle \in C^\infty(M)$

Q: Why is h a (0,2)-tensor?

Note: Let $F = \iota : M \hookrightarrow \mathbb{R}^n$

Check: $h(fX, Y) = f\,h(X,Y)$

and $h(X, fY) = f\,h(X,Y)$ ← $h(X, fY) = \langle D_X(fY), \nu \rangle$

$= \langle f D_X Y, \nu \rangle + \langle \cancelto{0}{X(f)Y, \nu} \rangle$

$= f \langle D_X Y, \nu \rangle$

GOAL: M^n C^∞-mfd ⇝

① exterior derivative d ✓ (on forms)
② Lie derivative L_X (on tensors)

§ Exterior derivative (Chern Ch.3)

Idea: ∇, div, curl ⟷ d

Notation: $A^k = \Gamma(\Lambda^k T^*M) = \Omega^k(M) = \{ k\text{-forms on } M \}$

"**Thm**": $\exists !$ exterior derivative

$$d = d_k : \Omega^k(M) \longrightarrow \Omega^{k+1}(M) \qquad \text{for } k \geq 0.$$

satisfying the following:

1) $df(X) = X(f)$ $\qquad \forall f \in \Omega^0(M) = C^\infty(\mathbb{R})$

2) $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$ $\qquad \forall \omega_1, \omega_2 \in \Omega^k(M)$.

3) $\boxed{d^2 = d \circ d = 0}$ (*) $\quad \left[\Longleftrightarrow \frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i} \quad \forall f \in C^\infty \right]$.

4) $d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^r \omega \wedge d\eta$ $\qquad \forall \omega \in \Omega^r(M), \eta \in \Omega^s(M)$

de Rham complex

$$\begin{tikzcd} C^\infty(M) = \Omega^0(M) \arrow[r, "d"] \arrow[rr, bend right, "d^2 = 0"'] & \Omega^1(M) \arrow[r, "d"] & \Omega^2(M) \arrow[r, "d"] & \cdots \arrow[r, "d"] & \Omega^n(M^n) \arrow[r, "d"] & 0 \end{tikzcd}$$

⟷ "de Rham cohomology"

Example: $\boxed{M^n = \mathbb{R}^n}$ $\qquad d : \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n)$

- $df = \sum a_i dx^i = \frac{\partial f}{\partial x^1} dx^1 + \cdots + \frac{\partial f}{\partial x^n} dx^n$

- $$d\left(\sum a_i dx^i\right) = \sum_{i=1}^n \left(d(a_i \wedge dx^i) \right) \qquad (2)$$

$$= \sum_{i=1}^n \left(\underset{\underset{\sum_{j=1}^n \frac{\partial a_i}{\partial x^j} dx^j}{\shortparallel}}{\underline{(da_i)}} \wedge dx^i + a_i \wedge \underbrace{d(dx^i)}_{d^2 x^i = 0 \text{ by } (3)} \right) \qquad (4)$$

$$= \sum_{1\le i<j\le n} \underbrace{\left(\frac{\partial a_j}{\partial x^i} - \frac{\partial a_i}{\partial x^j}\right)}_{\text{"curl"}} dx^i \wedge dx^j$$

Similarly, d is defined on Ω^k for all k.

Key Property: (diffeo. invariance) Given $F : \mathbb{R}^m \to \mathbb{R}^n$ smooth, and $\omega \in \Omega^k(\mathbb{R}^n)$, then $d(F^*\omega) = F^*(d\omega)$

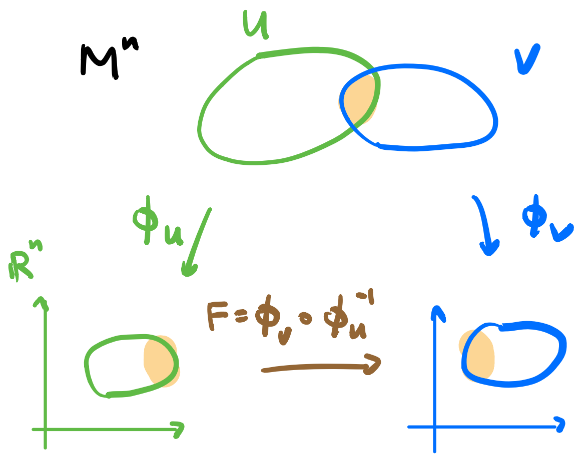
$$\boxed{d\circ F^* = F^*\circ d}$$

(Pf: Exercise)

- This allows us to define d on manifolds.

Why?

$\omega \in \Omega^k(M)$



$\omega_U \in \Omega^k(\mathbb{R}^n)$ $\quad$ $\omega_V \in \Omega^k(\mathbb{R}^n)$

$d\omega_U \in \Omega^{k+1}(\mathbb{R}^n)$ $\quad$ $d\omega_V \in \Omega^{k+1}(\mathbb{R}^n)$

Compatible: $\omega = \phi_U^*\omega_U = \phi_V^*\omega_V$

i.e. $\omega_U = F^*\omega_V = (\phi_V \circ \phi_U^{-1})^*\omega_V$

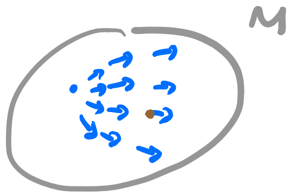
To define $d\omega$?

$$\begin{aligned} d\omega_U &= d(F^*\omega_V) \\ &= F^*(d\omega_V) \\ &= (\phi_V\circ\phi_U^{-1})^*(d\omega_V) \end{aligned}$$

ie $d\omega := \phi_U^*(d\omega_U) = \phi_V^*(d\omega_V)$.

§ Lie derivative (Chern § 6.2)

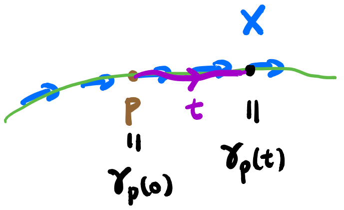
Idea: vector fields on M $\longleftrightarrow$ "infinitesimal" diffeomorphisms



Let $X \in \mathfrak{X}(M)$, at each $p \in M$, $\exists!$ integral curve $\gamma_p(t)$

(b) s.t. $\begin{cases} \gamma_p'(t) = X(\gamma_p(t)) & \text{for } t \in (-\varepsilon, \varepsilon) \\ \gamma_p(0) = p \end{cases}$

↑ autonomous ODE system.



Fix t $\quad \varphi_t : M \to M \qquad \varphi_t(p) = \gamma_p(t)$ as long as it's defined

(b) $\Rightarrow$ Property: $\varphi_{t+s} = \varphi_t \circ \varphi_s = \varphi_s \circ \varphi_t$

In particular, we have $\varphi_0 = id_M$ and $(\varphi_t)^{-1} = \varphi_{-t}$

So, $X \in \mathfrak{X}(M) \rightsquigarrow$ 1-parameter family $\{\varphi_t\}_t \subseteq \mathcal{D}iff(M) := \left\{ \begin{array}{l} F: M \to M \\ \text{diffeo.} \end{array} \right\}$

flow generated by X

Idea: Given $X \in \mathfrak{X}(M) \rightsquigarrow \{\varphi_t\} \subseteq \mathcal{D}iff(M)$

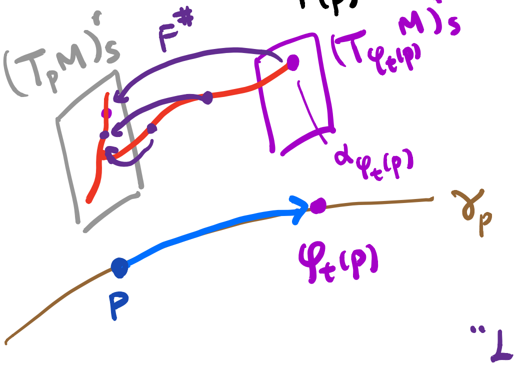
Recall: $F: M \to M$ diffeo. [Take $F = \varphi_t$]

$$\omega \in \Omega^1(M) \longrightarrow F^*\omega \in \Omega^1(M)$$

$$Y \in \mathfrak{X}(M) \longrightarrow (F^{-1})_* Y \in \mathfrak{X}(M)$$

Similarly, define $F^\# : \mathcal{T}(T^r_s M) \longrightarrow \mathcal{T}(T^r_s M)$ ← "pullback of (r,s)-tensors"

Note: $F^\#_{F(p)} : (T_{F(p)}M)^r_s \longrightarrow (T_pM)^r_s$ at $p \in M$.



Given $\alpha \in \mathcal{T}(T^r_s M)$,

$$t \longmapsto \varphi_t^\#(\alpha)\,(p)$$

is a smooth curve in $\underbrace{(T_pM)^r_s}_{\text{one fixed vector space}}$

"Lie derivative along X"

Def$^{\underline{n}}$: $\mathcal{L}_X \alpha := \left.\dfrac{d}{dt}\right|_{t=0} (\varphi_t^\# \alpha)$

Note: $d : \Omega^k \to \Omega^{k+1}$
$\mathcal{L}_X : \mathcal{T}(T^r_s M) \to \mathcal{T}(T^r_s M)$

Properties of $\mathcal{L}_X$ (on tensors) Fix $X \in \mathfrak{X}(M)$.

1) $\mathcal{L}_X f = X(f) \qquad \forall f \in C^\infty(M)$

2) $\mathcal{L}_X Y = [X, Y] \qquad \forall Y \in \mathfrak{X}(M)$

3) $\mathcal{L}_X(\alpha \otimes \beta) = (\mathcal{L}_X \alpha) \otimes \beta + \alpha \otimes (\mathcal{L}_X \beta)$

4) $\mathcal{L}_X \circ C = C \circ \mathcal{L}_X \quad$ where $C : \mathcal{T}(T^r_s M) \to \mathcal{T}(T^{r-1}_{s-1} M)$ contraction

Note: 1) – 3) ⇒ L_X well-defined on $\mathcal{T}(T^r_0 M)$

& 4) ⇒ L_X well-defined on $\mathcal{T}(T^r_s M)$

Example: $\omega \in \Omega^1(M) = \mathcal{T}(T^0_1 M)$. $\quad L_X\omega = ?$

(0,1)-tensor

Take any $Y \in \mathfrak{X}(M)$,

$$c\left(L_X(\omega\otimes Y) \overset{③}{=} (L_X\omega)\otimes Y + \omega\otimes(\underbrace{L_X Y}_{[X,Y]}) \; ② \right)$$

$$c\circ L_X(\omega\otimes Y) = (L_X\omega)(Y) + \omega([X,Y])$$

④ $\parallel$

$$L_X\circ c\,(\omega\otimes Y) = L_X(\omega(Y)) \overset{①}{=} X(\overset{\text{function}}{\omega(Y)})$$

$$\Rightarrow \quad \boxed{(L_X\omega)(Y) = X(\omega(Y)) - \omega([X,Y])}$$

Proof: 1) trivial ; 3) same proof as usual Leibniz rule

4) Claim: $c\circ L_X = L_X\circ c$.

Take any (r,s)-tensor α.

$$c\circ L_X\alpha = c\left(\lim_{t\to 0}\frac{\varphi_t^\#\alpha - \alpha}{t}\right)$$

$$= \lim_{t\to 0} c\left(\frac{\varphi_t^\#\alpha - \alpha}{t}\right)$$

$$= \lim_{t\to 0}\frac{c(\varphi_t^\#\alpha) - c(\alpha)}{t}$$

$$\overset{(*)}{=} \lim_{t\to 0}\frac{\varphi_t^\#(c\alpha) - c\alpha}{t} = L_X(c\alpha)$$

(*) true if $\boxed{c\circ\varphi_t^\# = \varphi_t^\#\circ c}$

why? E.g.) $\alpha = Y\otimes\omega$ (1,1) tensor ; let $F = \varphi_t$

$$c\circ F^\#(\alpha) = c\left((F^{-1})_*Y \otimes F^*\omega\right) = (F^*\omega)\left((F^{-1})_*X\right)$$

$$= F^*\left[\omega(\underbrace{F_*F_*^{-1}}_{id}X)\right] = F^*[\omega(X)] = F^\#\circ c(\alpha)$$

2) Observe : (Recall : $L_X Y = \lim_{t\to 0} \frac{\varphi_t^{\#} Y - Y}{t} = \lim_{t\to 0} \frac{\varphi_{t*}^{-1} Y - Y}{t}$)

$$(L_X Y)_p = \lim_{t\to 0} \frac{\varphi_{t*}^{-1}(Y_{\varphi_t(p)}) - Y_p}{t} = \lim_{t\to 0} \frac{Y_p - \varphi_{t*}(Y_{\varphi_t^{-1}(p)})}{t} .$$

$\|$

$[X,Y](p)$

$\|$

$(XY - YX)(p)$

Take $f \in C^\infty(M)$.

$$\frac{Y_p - \varphi_{t*}(Y_{\varphi_t^{-1}(p)})}{t}(f) \xrightarrow{\text{as } t\to 0} (L_X Y)_p(f)$$

$$= \frac{Y_p(f) - Y_{\varphi_t^{-1}(p)}(f\circ\varphi_t)}{t}$$

$$= \underbrace{\left(\frac{Y_p(f) - Y_{\varphi_t^{-1}(p)}(f)}{t}\right)}_{\text{as } t\to 0,\ \to\ X_p(Y(f))} - \underbrace{\left(\frac{Y_{\varphi_t^{-1}(p)}(f\circ\varphi_t) - Y_{\varphi_t^{-1}(p)}(f)}{t}\right)}_{\text{as } t\to 0,\ \searrow\ Y_p(X(f))}$$

$$= \mathrm{I} + \mathrm{II}$$

$$\mathrm{I} = \frac{Y(f)_p - Y(f)_{\varphi_t^{-1}(p)}}{t} \longrightarrow X_p(Y(f))$$

$$\mathrm{II} = \underbrace{Y_{\varphi_t^{-1}(p)}}_{\downarrow\ Y_p} \left[\underbrace{\frac{f\circ\varphi_t - f}{t}}_{\searrow\ X(f)} \right] \longrightarrow Y_p(X(f)).$$

——— □